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2004 J. Phys. A: Math. Gen. 37 11531

(http://iopscience.iop.org/0305-4470/37/48/002)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 11531–11546 PII: S0305-4470(04)80391-3

Global existence and nonexistence results for a generalized Davey–Stewartson system

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Received 6 May 2004, in final form 14 September 2004
Published 17 November 2004
Online at stacks.iop.org/JPhysA/37/11531
doi:10.1088/0305-4470/37/48/002

Abstract
We consider a system of three equations, which will be called generalized Davey–Stewartson equations, involving three coupled equations, two for the long waves and one for the short wave propagating in an infinite elastic medium. We classify the system according to the signs of the parameters. Conserved quantities related to mass, momentum and energy are derived as well as a specific instance of the so-called virial theorem. Using these conservation laws and the virial theorem both global existence and nonexistence results are established under different constraints on the parameters in the elliptic–elliptic–elliptic case.

PACS numbers: 02.30.Jr, 62.30.+d

1. Introduction

In a recent study, Babaoglu and Erbay, using a multi-scale expansion of quasi-monochromatic wave solutions, derived a system of three nonlinear equations to model wave propagation in a bulk medium composed of an elastic material with couple stresses [1]. In terms of dimensionless variables the equations read [1] (equation (26))

$$\begin{aligned}
&\mathrm{i}u_t + \delta u_{xx} + u_{yy} = \chi |u|^2 u + b(\varphi_{1,x} + \varphi_{2,y})u, \\
&\varphi_{1,xx} + m_2\varphi_{1,yy} + n\varphi_{2,xy} = (|u|^2)_x, \\
&\lambda\varphi_{2,xx} + m_1\varphi_{2,yy} + n\varphi_{1,xy} = (|u|^2)_y,
\end{aligned} \tag{1}$$

where u is the scaled complex amplitude of the free short transverse wave mode and φ_1 and φ_2 are the scaled-free long longitudinal and long transverse wave modes, respectively. These nonlinear equations may be called a generalized Davey–Stewartson (GDS) equations. A fundamental role is played by the relation

$$(\lambda - 1)(m_2 - m_1) = n^2, \tag{2}$$

where $\lambda > 1$ and $m_2 > m_1$. The parameters χ and b can be of any sign but $\delta > 0$ (see section 3 for their explicit expressions).

It was shown in [1] that if $n = 1 - \lambda = m_1 - m_2$ then equations (1) can be transformed into the classical Davey–Stewartson (DS) equations [2, 3] via a nonlinear transformation

$$\mathrm{i}u_t + \delta u_{xx} + u_{yy} = (\chi + b)|u|^2 u + bQu, \qquad Q_{xx} + m_1 Q_{yy} = (1 - m_1)(|u|^2)_{yy}, \tag{3}$$

where $Q = \varphi_{1,x} + \varphi_{2,y} - |u|^2$. It is therefore natural to ask: 'to what extent does the qualitative behaviour of the GDS system mimic that of the DS system?'

Under some constraints on physical parameters χ, b, λ, m_1 and m_2, we have observed that the analogy is remarkable and the present study aims to develop this analogy. In fact, in the elliptic–elliptic–elliptic case we will prove a theorem on global nonexistence solutions of the GDS system as well as a global existence result which is in agreement with Ghidaglia–Saut result in [4] (compare theorem 3.2 [4] with our theorem 6.1).

In the literature, apart from extending $N = 2$ to arbitrary dimensions and changing $(3)_2$ to $Q_{xx} + m_1 Q_{yy} = \left(1 + \frac{1}{m_1}\right)(|u|^2)_{xx}$, various generalizations to DS system have been proposed for modelling the evolution of weakly nonlinear waves. Here we will only cite a few examples in order to compare their structure to GDS system (1). Zakharov–Schulman have studied systems of the form

$$\mathrm{i}A_t + L_1 A + A\Psi = 0, \qquad L_2\Psi = L_3|A|^2$$

where L_1, L_2 and L_3 are second-order differential operators with constant coefficients [5, 6]. Grimshaw [7], on the other hand, considered

$$\mathrm{i}A_t + \lambda_1 A_{xx} + \lambda_2 A_{yy} = \gamma|A|^2 A + SA$$

where $S = S_1 + S_2$ with

$$\Delta S_2 = \nu_0(|A|^2)_{yy}, \qquad S_1 = \sum_{s=1}^{\infty} \delta_s W_s, \qquad \Delta W_s - \alpha^2 W_{s,xx} = \tilde{\nu}(|A|^2)_{yy}.$$

The system considered by Oikawa, Chow and Benney [8] is in some sense closest to our system; it is given by

$$\begin{aligned}
&\mathrm{i}A_t + \beta_1 A_{xx} + \beta_2 A_{zz} + \gamma_0|A|^2 A + (\gamma_1 L_1 + \gamma_2 L_2 + \gamma_3 L_3)A = 0,\\
&\alpha L_{1,xx} + L_{1,zz} = \delta_1(|A|^2)_{xx} + \delta_2(|A|^2)_{zz},\\
&L_{2,xx} = L_{1,xx} + L_{1,zz}, \qquad L_{3,xx} = (|A|^2)_{zz}.
\end{aligned}$$

Here A is the amplitude of the wave packet and L_1, L_2 and L_3 measure the mean flow effects. To contrast our model with the above-mentioned models, we note that in the Zakharov–Schulman model mean flow is only affected by one variable A. In the second model, both mean flow effects S_1 and S_2 depend only on $(|A|^2)_{yy}$, and finally, in the last model the variables L_1, L_2 and L_3 do not all interact; rather L_1 determines L_2, and $|A|^2$ determines both L_1 and L_3. In contrast in our case, it is precisely the nature of the interaction between the two mean flow variables φ_1 and φ_2 in (1) that makes the model worth studying.

Our paper is organized as follows: in section 2, we start by recalling the previous results for DS equations that are relevant to the present study. In section 3, explicit description of the parameters is introduced and the essential relation (2) is derived. Then depending on the signs of m_1, m_2 and λ, we classify the GDS system as elliptic–elliptic–elliptic, elliptic–elliptic–hyperbolic and elliptic–hyperbolic–hyperbolic. Similarly to those of DS equations [4], our global existence and nonexistence results hold in the elliptic–elliptic–elliptic case. In section 4 we derive four conserved quantities for mass (17), momentum (20) and energy (23). Section 5 is devoted to a global existence result when $m_1 > 0$ and $\chi \geqslant \max\left\{-b\max\left(1, \frac{1}{m_1}\right), 0\right\}$ for solutions that exist in $H^1(\mathbb{R}^2)$ locally in time. As a direct corollary, the stability of small amplitude solutions is established. In section 6, we start by establishing a specific instance of the virial theorem (see Chandrasekhar [9])

$$\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = 8E(u(t)). \tag{4}$$

(In the Hamiltonian formulation, the Hamiltonian can be seen to be equivalent to E). This fundamental relation allows us to conclude that when $\chi < \min\left(-\frac{b}{m_1}, 0\right)$ and $m_1 > 1$ solutions with $E(u_0) < 0$ cannot exist for arbitrary finite time. This result also allows us to deduce an instability result for nonzero ground state solutions. Finally, we end the paper with some remarks on existence and uniqueness of solutions and on the possibility of extending solutions. These remarks, we hope, will give some justifications for the formal computations in the next four sections.

2. Background

The Davey–Stewartson system is a model for the evolution of weakly nonlinear packets of water waves that travel predominantly in one direction but in which the amplitude of waves is modulated in two spatial directions. They are given as

$$\mathrm{i}u_t + \delta u_{xx} + u_{yy} = \chi |u|^2 u + bu\varphi_x, \qquad \varphi_{xx} + m\varphi_{yy} = (|u|^2)_x, \tag{5}$$

where u is the complex amplitude of the short wave and φ is the real long wave amplitude [2, 3]. The literature on DS equations is quite extensive; here we have only tried to cite those works that we have been inspired by. The physical parameters δ and m play a determining role in the classification of this system. Depending on their signs, the system is elliptic–elliptic, elliptic–hyperbolic and hyperbolic–elliptic [4]. For the elliptic–elliptic case in [4] it is proven that the solution of DS system (5) that exists for finite time is global if $\chi \geqslant \max(-b, 0)$ and the solution cannot exist globally in time if $\chi < \max(-b, 0)$. The primary role in the argument is played by the conserved quantities $M(u)$ and $E(u)$, whereas the conservation of momentum $J_x(u)$ and $J_y(u)$ plays a secondary role:

$$\begin{aligned}
M(u) &= \int_{\mathbb{R}^2} |u|^2 \,\mathrm{d}x\,\mathrm{d}y, \\
J_x(u) &= \int_{\mathbb{R}^2} (uu_x^* - u^* u_x)\,\mathrm{d}x\,\mathrm{d}y, \\
J_y(u) &= \int_{\mathbb{R}^2} (uu_y^* - u^* u_y)\,\mathrm{d}x\,\mathrm{d}y, \\
E(u) &= \int_{\mathbb{R}^2} \left\{ \delta |u_x|^2 + |u_y|^2 + \frac{\chi}{2}|u|^4 + \frac{b}{2}[(\varphi_x)^2 + m(\varphi_y)^2] \right\} \mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

Ablowitz and Segur [10], when considering the focusing effect, have established a version of virial theorem, i.e.

$$\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = 8E(u), \tag{6}$$

where $I(t) = \int_{\mathbb{R}^2} \left(\frac{1}{\delta}x^2 + y^2\right)|u|^2 \,\mathrm{d}x\,\mathrm{d}y$. In this paper, focusing and global nonexistence will be used interchangeably.

3. On classification of the GDS system

The generalized Davey–Stewartson system involving three coupled nonlinear equations (1) has been introduced [1] to study (2+1)-dimensional waves in a bulk elastic medium. They are given by

$$\begin{aligned} &\mathrm{i}A_\tau + pA_{\xi\xi} + rA_{\eta\eta} = q|A|^2A + \frac{k^2}{2\omega}(\gamma_3\phi_{1,\xi} + \gamma_1\phi_{2,\eta})A, \\ &\left(c_g^2 - c_1^2\right)\phi_{1,\xi\xi} - c_2^2\phi_{1,\eta\eta} - \left(c_1^2 - c_2^2\right)\phi_{2,\xi\eta} = \gamma_3 k^2(|A|^2)_\xi, \\ &\left(c_g^2 - c_2^2\right)\phi_{2,\xi\xi} - c_1^2\phi_{2,\eta\eta} - \left(c_1^2 - c_2^2\right)\phi_{1,\xi\eta} = \gamma_1 k^2(|A|^2)_\eta, \end{aligned} \tag{7}$$

where $c_g = c_2^2(k + 8m^2k^3)/\omega$ and $\omega = c_2k(1 + 4m^2k^2)^{\frac{1}{2}}$, and

$$\begin{aligned} &c_1^2 = \frac{\tilde{\lambda} + 2\tilde{\mu}}{\rho_0}, \qquad c_2^2 = \frac{\tilde{\mu}}{\rho_0}, \\ &\gamma_1 = c_1^2 - 2c_2^2 + \frac{\mathcal{B}}{\rho_0}, \qquad \gamma_3 = c_1^2 + \frac{\mathcal{A} + 2\mathcal{B}}{2\rho_0}, \\ &p = -\frac{1}{2\omega}\left(c_g^2 - c_2^2 - 24m^2c_2^2k^2\right), \qquad r = \frac{c_2^2}{2\omega}(1 + 8m^2k^2), \qquad q = \frac{k^6\gamma_3^2}{\omega D_1(2k, 2\omega)}. \end{aligned} \tag{8}$$

Here k is the wave number, ω is the frequency and m, $\tilde{\lambda}$, $\tilde{\mu}$, $\mathcal{A}$ and $\mathcal{B}$ are material constants; c_g is the group speed of transverse waves whereas c_1 and c_2 are phase speeds of longitudinal and transverse waves, respectively. In terms of dimensionless variables, the GDS system takes the form

$$\begin{aligned} &\mathrm{i}u_t + \delta u_{xx} + u_{yy} = \chi|u|^2u + b(\varphi_{1,x} + \varphi_{2,y})u, \\ &\varphi_{1,xx} + m_2\varphi_{1,yy} + n\varphi_{2,xy} = (|u|^2)_x, \\ &\lambda\varphi_{2,xx} + m_1\varphi_{2,yy} + n\varphi_{1,xy} = (|u|^2)_y, \end{aligned} \tag{9}$$

where the non-dimensional coefficients that play the key role in the classification of the system are given by

$$\begin{aligned} &m_1 = \frac{c_1^2}{c_1^2 - c_g^2}\left(\frac{\gamma_3}{\gamma_1}\right)^2, \qquad m_2 = \frac{c_2^2}{c_1^2 - c_g^2}\left(\frac{\gamma_3}{\gamma_1}\right)^2, \\ &\lambda = \frac{c_2^2 - c_g^2}{c_1^2 - c_g^2}, \qquad n = \frac{c_1^2 - c_2^2}{c_1^2 - c_g^2}\left(\frac{\gamma_3}{\gamma_1}\right), \end{aligned} \tag{10}$$

so $(\lambda - 1)(m_2 - m_1) = n^2$. A simple algebra shows that $c_g^2 - c_2^2 = 4m^2c_2^4k^4(3 + 16m^2k^2)/\omega^2$ is always positive. However, it is possible to show that there exists a critical wave number $k_c^2 = \left[c_1^2 - 4c_2^2 + c_1\left(c_1^2 + 8c_2^2\right)^{\frac{1}{2}}\right]/\left(32c_2^2m^2\right)$ such that $c_1^2 - c_g^2 < 0$ if $k > k_c$ and $c_1^2 - c_g^2 > 0$ if $k < k_c$ because $c_1^2 > c_2^2$. (The case where $k = k_c$, corresponds to long-wave short-wave resonance since the phase speed of the longitudinal wave, c_1, is equal to the group speed of the transverse wave, c_g. Mathematically, this case corresponds to a degenerate case similar to that of (2+1) nonlinear differential equations for DS [11].) Thus, depending on the wave number k chosen, the coefficients of the second and third equations of the GDS system may change their sign. For example, the respective sign of (m_1, m_2, λ) is $(-, -, +)$ if $k > k_c$ and is $(+, +, -)$ if $k < k_c$. A simple algebra shows that p in (8) is always positive, so is δ, henceforth we will set $\delta = 1$ without loss of generality.

Now the GDS system will be classified according to the values of parameters. In fact, since $\delta > 0$ the first equation is always elliptic, so we will only consider the last two coupled equations of GDS system (9) involving the variables φ_1 and φ_2, and write their leading order linear part as a first-order linear system

$$\mathbf{A}\mathbf{v}_x + \mathbf{B}\mathbf{v}_y = \mathbf{0}, \tag{11}$$

where

$$\mathbf{A} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & m_2 & n & 0 \\ n & 0 & 0 & m_1 \end{pmatrix}, \tag{12}$$

in which $\mathbf{v} = (v_1, v_2, v_3, v_4)^T$ and where

$$\varphi_{1,x} = v_1, \qquad \varphi_{1,y} = v_2, \qquad \varphi_{2,x} = v_3, \qquad \varphi_{2,y} = v_4.$$

Using $(\lambda - 1)(m_2 - m_1) = n^2$, the eigenvalues of $\mathbf{A}^{-1}\mathbf{B}$ are then expressed in terms of the parameters m_1, m_2 and λ as

$$r_1 = -\left(-\frac{m_2}{\lambda}\right)^{\frac{1}{2}}, \qquad r_2 = \left(-\frac{m_2}{\lambda}\right)^{\frac{1}{2}}, \qquad r_3 = -(-m_1)^{\frac{1}{2}}, \qquad r_4 = (-m_1)^{\frac{1}{2}}.$$

Therefore, system (1) can be classified as elliptic–elliptic–elliptic, elliptic–hyperbolic–hyperbolic, and elliptic–elliptic–hyperbolic according to the respective sign of (m_1, m_2, λ): $(+, +, +)$, $(-, -, +)$ and $(+, +, -)$. Depending on the value of the wave number k, it has already been shown in this section that the last two cases correspond to two different physical cases. The critical wave number k_c depends on material properties. Using the data given in Erofeyev and Potapov [12] we have computed $k_c = 1794$ for Brass LS-62 and $k_c = 2114$ for Bronze BROF 35. At $k = k_c$ the equations change type; namely the elliptic–hyperbolic–hyperbolic case appears when $k > k_c$ and the elliptic–elliptic–hyperbolic case appears when $k < k_c$, whereas $k = k_c$ corresponds to the long-wave short-wave resonant case where the phase speed of the longitudinal wave and the group speed of the transverse wave are equal.

4. Conservation laws

In this section we derive four conservation laws, namely (17), (20) and (23), for the GDS system. These conservation laws were derived first by direct computation; however, here we will give an alternative derivation that follows from the invariance of the Lagrangian relative to infinitesimal transformations.

The Lagrangian density function of the GDS system (1) is given by

$$\mathcal{L} = \frac{i}{2}(u^* u_t - u u_t^*) - \frac{\chi}{2}|u|^4 - |u_x|^2 - |u_y|^2 - b(\varphi_{1,x} + \varphi_{2,y})|u|^2 \\ + \frac{b}{2}[(\varphi_{1,x})^2 + m_2(\varphi_{1,y})^2 + \lambda(\varphi_{2,x})^2 + m_1(\varphi_{2,y})^2 + n(\varphi_{1,y}\varphi_{2,x} + \varphi_{1,x}\varphi_{2,y})]. \tag{13}$$

According to Noether's theorem [13–15], if the functional

$$S\{u, \varphi_1, \varphi_2\} = \int \mathcal{L}(u, u^*, u_t, u_t^*, u_x, u_x^*, u_y, u_y^*, \varphi_{1,x}, \varphi_{1,y}, \varphi_{2,x}, \varphi_{2,y})\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}t,$$

is invariant under the transformations

$$\begin{aligned} x_j' &= x_j + \delta x_j(t, x, y, u, u^*, \ldots) \qquad (j = 0, 1, 2) \\ \psi_i' &= \psi_i + \delta\psi_i(t, x, y, u, u^*, \ldots) \qquad (i = 1, 2, 3, 4) \end{aligned} \tag{14}$$

where ($x_0 = t, x_1 = x, x_2 = y$) and ($\psi_1 = u, \psi_2 = u^*, \psi_3 = \varphi_1, \psi_4 = \varphi_2$), and δx_j and $\delta\psi_i$ are proportional to infinitesimal ϵ, then

$$\frac{\partial I_j}{\partial x_j} = 0, \tag{15}$$

with

$$I_j = \frac{\partial \mathcal{L}}{\partial(\partial\psi_i/\partial x_j)}\left(\frac{\partial\psi_i}{\partial x_k}\delta x_k - \delta\psi_i\right) - \mathcal{L}\delta x_j \qquad (j, k = 0, 1, 2, i = 1, 2, 3, 4).$$

Here summation convention is valid over repeated indices and u^* is the complex conjugate of u. Assuming that $u \in H^1(\mathbb{R}^2)$ and $\nabla\varphi_1, \nabla\varphi_2 \in \mathbb{L}^2(\mathbb{R}^2)$, integration of the continuity equation (15) over $\mathbb{R}^2$ yields the conservation law

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2} I_0 \,\mathrm{d}x\,\mathrm{d}y = 0.$$

Invariance by phase shift. The Lagrangian (13) is invariant by the global transformation $u' = \mathrm{e}^{\mathrm{i}\epsilon}u$, i.e. $u' \simeq u + \mathrm{i}\epsilon u$ where $\delta t = \delta x = \delta y = \delta\varphi_1 = \delta\varphi_2 = 0$ and $\delta u = \mathrm{i}\epsilon u$. Equation (15) then takes the form

$$(|u|^2)_t + [\mathrm{i}(uu_x^* - u^*u_x)]_x + [\mathrm{i}(uu_y^* - u^*u_y)]_y = 0. \tag{16}$$

This leads to the conservation of mass,

$$M = \int_{\mathbb{R}^2} |u|^2 \,\mathrm{d}x\,\mathrm{d}y. \tag{17}$$

Invariance by space translations. The Lagrangian (13) is invariant by the infinitesimal space translation $x' = x + \delta x$ with $\delta t = \delta y = \delta u = \delta\varphi_1 = \delta\varphi_2 = 0$. In this case, equation (15) becomes

$$\begin{aligned}
&\mathrm{i}(u^*u_x - uu_x^*)_t + \{-2|u_x|^2 + 2|u_y|^2 + \chi|u|^4 - \mathrm{i}(u^*u_t - uu_t^*)\\
&\quad + b[(\varphi_{1,x})^2 + \lambda(\varphi_{2,x})^2 - m_2(\varphi_{1,y})^2 - m_1(\varphi_{2,y})^2 + 2\varphi_{2,y}|u|^2]\}_x\\
&\quad + 2\{-u_xu_y^* - u_x^*u_y + b\varphi_{1,x}(m_2\varphi_{1,y} + n\varphi_{2,x}) + b\varphi_{2,x}(-|u|^2 + m_1\varphi_{2,y})\}_y = 0.
\end{aligned} \tag{18}$$

Similarly the space translation $y' = y + \delta y$ leads to

$$\begin{aligned}
&\mathrm{i}(u^*u_y - uu_y^*)_t + 2\{-u_x^*u_y - u_xu_y^* + b\varphi_{1,y}(-|u|^2 + \varphi_{1,x}) + b\varphi_{2,y}(\lambda\varphi_{2,x} + n\varphi_{1,y})\}_x\\
&\quad + \{-2|u_y|^2 + 2|u_x|^2 + \chi|u|^4 - \mathrm{i}(u^*u_t - uu_t^*) + b[m_2(\varphi_{1,y})^2 + m_1(\varphi_{2,y})^2\\
&\quad - (\varphi_{1,x})^2 - \lambda(\varphi_{2,x})^2 + 2\varphi_{1,x}|u|^2]\}_y = 0.
\end{aligned} \tag{19}$$

The conservation laws (18) and (19) lead to the conservation of momentum

$$\begin{aligned}
J_x &= \mathrm{i}\int_{\mathbb{R}^2} (u^*u_x - uu_x^*)\,\mathrm{d}x\,\mathrm{d}y,\\
J_y &= \mathrm{i}\int_{\mathbb{R}^2} (u^*u_y - uu_y^*)\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned} \tag{20}$$

Invariance by time translation. The Lagrangian (13) is invariant by the infinitesimal time translation $t' = t + \delta t$ with $\delta x = \delta y = \delta u = \delta\varphi_1 = \delta\varphi_2 = 0$. In this case, equation (15) becomes

$$\Big\{|u_x|^2 + |u_y|^2 + \frac{\chi}{2}|u|^4 + b(\varphi_{1,x} + \varphi_{2,y})|u|^2 - \frac{b}{2}[(\varphi_{1,x})^2 + m_2(\varphi_{1,y})^2 + \lambda(\varphi_{2,x})^2$$
$$+ m_1(\varphi_{2,y})^2 + n(\varphi_{1,y}\varphi_{2,x} + \varphi_{1,x}\varphi_{2,y})]\Big\}_t + \Big\{-(u_x^* u_t + u_x u_t^*)$$
$$+ b\varphi_{1,t}\left(-|u|^2 + \varphi_{1,x} + \frac{n}{2}\varphi_{2,y}\right) + b\varphi_{2,t}\left(\lambda\varphi_{2,x} + \frac{n}{2}\varphi_{1,y}\right)\Big\}_x$$
$$+ \Big\{-(u_y^* u_t + u_y u_t^*) + b\varphi_{1,t}\left(m_2\varphi_{1,y} + \frac{n}{2}\varphi_{2,x}\right)$$
$$+ b\varphi_{2,t}\left(-|u|^2 + m_1\varphi_{2,y} + \frac{n}{2}\varphi_{1,x}\right)\Big\}_y = 0. \tag{21}$$

This conservation law leads to the conservation of energy, i.e. the Hamiltonian

$$H = \int_{\mathbb{R}^2} \mathcal{H}\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}^2} \Big\{|u_x|^2 + |u_y|^2 + \frac{\chi}{2}|u|^4 + b(\varphi_{1,x} + \varphi_{2,y})|u|^2 - \frac{b}{2}[(\varphi_{1,x})^2$$
$$+ m_2(\varphi_{1,y})^2 + \lambda(\varphi_{2,x})^2 + m_1(\varphi_{2,y})^2 + n(\varphi_{1,y}\varphi_{2,x} + \varphi_{1,x}\varphi_{2,y})]\Big\}\,\mathrm{d}x\,\mathrm{d}y, \tag{22}$$

where $\mathcal{H}$ is the Hamiltonian density function. It will be sometimes more convenient to rewrite the Hamiltonian as follows:

$$H = \int_{\mathbb{R}^2} \Big\{|u_x|^2 + |u_y|^2 + \frac{\chi}{2}|u|^4 + \frac{b}{2}[(\varphi_{1,x})^2 + m_2(\varphi_{1,y})^2 + \lambda(\varphi_{2,x})^2$$
$$+ m_1(\varphi_{2,y})^2 + n(\varphi_{1,y}\varphi_{2,x} + \varphi_{1,x}\varphi_{2,y})]\Big\}\,\mathrm{d}x\,\mathrm{d}y, \tag{23}$$

which is obtained from (22) by integration by parts and utilizing $(1)_2$ and $(1)_3$. We remark that the conservation laws (17) and (23) in conjunction with the virial theorem will be crucial in the proof of the global nonexistence result in the sixth section.

5. Global existence of solutions and stability of small amplitude solutions

In this work, we do not completely resolve the question of local in time existence of solutions of the GDS system in the elliptic–elliptic–elliptic case. However, assuming that a unique solution u exists in $H^1(\mathbb{R}^2)$ and it can be continued as long as its H^1-norm remains bounded we will prove the global existence of the solution of the GDS system in the elliptic–elliptic–elliptic case where m_1, m_2 and λ are positive.

Theorem 5.1. *Suppose that* $m_1 > 0$ *and* $\chi \geqslant \max\left\{-b\max\left(1, \frac{1}{m_1}\right), 0\right\}$*, then the solutions of the GDS system (1) are global:* $T = \infty$.

Proof. Using the Plancherel theorem, the energy (Hamiltonian) (23) of the solution u of (1) is expressed as

$$H(u) = \int \Big\{\left(\xi_1^2 + \xi_2^2\right)|\hat{u}|^2 + \frac{\chi}{2}|\hat{f}|^2 + \frac{b}{2}\big[\xi_1^2|\hat{\varphi}_1|^2 + m_2\xi_2^2|\hat{\varphi}_1|^2$$
$$+ \lambda\xi_1^2|\hat{\varphi}_2|^2 + m_1\xi_2^2|\hat{\varphi}_2|^2 + 2n\xi_1\xi_2\hat{\varphi}_1\hat{\varphi}_2^*\big]\Big\}\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2 \tag{24}$$

where (ξ_1, ξ_2) is the dual variable of (x, y), $\hat{u}$ is the Fourier transform of u and $\hat{f}$ is the Fourier transform of $|u|^2$. Calculating the Fourier transforms of the second and third equations of (1) enables us to express the transformed variables $\hat{\varphi}_1$ and $\hat{\varphi}_2$ in terms of $\hat{f}$:

$$\hat{\varphi}_1 = \frac{\mathrm{i}\xi_1}{\Delta}\left(n\xi_2^2 - \lambda\xi_1^2 - m_1\xi_2^2\right)\hat{f},$$
$$\hat{\varphi}_2 = \frac{\mathrm{i}\xi_2}{\Delta}\left(n\xi_1^2 - \xi_1^2 - m_2\xi_2^2\right)\hat{f}, \tag{25}$$

where

$$\Delta = \lambda\xi_1^4 + (m_1 + \lambda m_2 - n^2)\xi_1^2\xi_2^2 + m_1 m_2\xi_2^2$$
$$= \left(\lambda\xi_1^2 + m_2\xi_2^2\right)\left(\xi_1^2 + m_1\xi_2^2\right),$$

since $m_1 + \lambda m_2 - n^2 = \lambda m_1 + m_2$ follows from (2). Thus in the elliptic–elliptic–elliptic case $\Delta > 0$ for all $(\xi_1, \xi_2) \neq (0, 0)$. Substituting (25) into the Hamiltonian (24) gives

$$H(u) = \int \left(\xi_1^2 + \xi_2^2\right)|\hat{u}|^2\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2 + \frac{1}{2}\int (\chi + \alpha b)|\hat{f}|^2\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2 \tag{26}$$

where

$$\alpha = \alpha(\xi_1, \xi_2) = \frac{\lambda\xi_1^4 + (1 + m_1 - 2n)\xi_1^2\xi_2^2 + m_2\xi_2^4}{\left(\lambda\xi_1^2 + m_2\xi_2^2\right)\left(\xi_1^2 + m_1\xi_2^2\right)}. \tag{27}$$

It is possible to show that $0 < \frac{1}{m_1} \leqslant \alpha \leqslant 1$ for $m_1 \geqslant 1$ and $1 \leqslant \alpha \leqslant \frac{1}{m_1}$ for $0 < m_1 < 1$ (see appendix A). Thus, if $\chi \geqslant \max\left\{-b\max\left(1, \frac{1}{m_1}\right), 0\right\}$, then $\chi + \alpha b \geqslant 0$ for both positive and negative values of b. In such a case, the energy (26) is bounded below by

$$H(u) \geqslant \int \left(\xi_1^2 + \xi_2^2\right)|\hat{u}|^2\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2,$$
$$= \int_{\mathbb{R}^2} (|u_x|^2 + |u_y|^2)\,\mathrm{d}x\,\mathrm{d}y. \tag{28}$$

Thus, the conservation of mass (17) and energy (28) lead to a uniform bound on the H^1 norm of u stating that the solution is global if it exists locally. □

A consequence of (28) is the following stability result of the solution $u = \varphi_1 = \varphi_2 = 0$.

Lemma 5.2. *Suppose that* $m_1 > 0$ *and* $\chi \geqslant \max\left\{-b\max\left(1, \frac{1}{m_1}\right), 0\right\}$*, then the zero solution of the GDS system (1) is stable.*

Proof. Assume that $\|u_0\|_{H^1} < \epsilon$ and $u(t)$ is the corresponding solution in the elliptic–elliptic–elliptic case, then $H(u(t)) = H(u_0)$ by (23). Since $m_1 > 0$ and $\chi \geqslant \max\left\{-b\max\left(1, \frac{1}{m_1}\right), 0\right\}$, then $\|\nabla u(t)\|_{L^2}^2 \leqslant H(u(t))$; therefore it suffices to bound $H(u_0)$ above. Using the mass conservation (17) and (26) we have

$$H(u_0) \leqslant \|u_0\|_{H^1}^2 + \frac{1}{2}\int (\chi + \alpha b)|\hat{f}_0|^2\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2. \tag{29}$$

When $m_1 > 0$ and $\chi \geqslant \max\left\{-b\max\left(1, \frac{1}{m_1}\right), 0\right\}$, $\chi + \alpha b \geqslant 0$ and we obtain

$$\|u(t)\|_{H^1}^2 \leqslant \|u_0\|_{L^2}^2 + H(u_0) < 2\epsilon^2 + \frac{1}{2}(|\chi| + \alpha_M|b|)\int_{\mathbb{R}^2} |u_0|^4\,\mathrm{d}x\,\mathrm{d}y$$
$$< 2\epsilon^2 + \frac{1}{2}(|\chi| + \alpha_M|b|)c^4\epsilon^4, \tag{30}$$

where $\alpha_M \equiv \max\left(1, \frac{1}{m_1}\right)$ and $\|u_0\|_{L^4} \leqslant c\|u_0\|_{H^1}$ by the Sobolev imbedding theorem. Given $\delta > 0$, one can choose $\epsilon > 0$ small enough so that the right-hand side of (30) is less than $\sqrt{\delta}$. □

6. Global nonexistence of solutions and instability of ground states

In this section we show that the solutions of the GDS system (1) in the elliptic–elliptic–elliptic case where m_1, m_2 and λ are positive, cannot exist globally in time. To this end, we use the *method of moments* (or the virial theorem) that is the classical approach to determine whether a given initial wave will collapse into a singular point in a finite time, i.e. the wave amplitude blows up at this point. The method was first developed by Vlassov *et al* [16] and applied to self-focusing phenomena in the NLS equation and to DS equations by Ablowitz and Segur [10]. Here we extend the method to the GDS system. For this aim, we introduce the quantity $I(t)$, the 'moment of inertia' of a localized waveform, whose evolution is related with the Hamiltonian (23) of system (1).

We present here the main steps in the derivation of the time evolution of the 'moment of inertia' of a localized solution of the GDS system. The 'moment of inertia' of a localized solution is defined as

$$I(t) = \int_{\mathbb{R}^2} (x^2 + y^2)|u|^2 \,\mathrm{d}x\,\mathrm{d}y. \tag{31}$$

By direct calculation, we find from equations (31) and (16):

$$\begin{aligned}\frac{\mathrm{d}I}{\mathrm{d}t} &= \int_{\mathbb{R}^2} (x^2 + y^2)(|u|^2)_t \,\mathrm{d}x\,\mathrm{d}y,\\ &= \mathrm{i}\int_{\mathbb{R}^2} (x^2 + y^2)[(u^*u_x - uu_x^*)_x + (u^*u_y - uu_y^*)_y]\,\mathrm{d}x\,\mathrm{d}y,\\ &= -2\mathrm{i}\int_{\mathbb{R}^2} [x(u^*u_x - uu_x^*) + y(u^*u_y - uu_y^*)]\,\mathrm{d}x\,\mathrm{d}y,\\ &= 4\,\mathrm{Im}\int_{\mathbb{R}^2} (xu^*u_x + yu^*u_y)\,\mathrm{d}x\,\mathrm{d}y,\end{aligned}$$

where $u \in H^1(\mathbb{R}^2)$ allowed us to use integration by parts. The second derivative of $I(t)$ is similarly obtained as

$$\frac{\mathrm{d}^2I}{\mathrm{d}t^2} = 8\,\mathrm{Im}\int_{\mathbb{R}^2} (xu_x + yu_y + u)u_t^* \,\mathrm{d}x\,\mathrm{d}y. \tag{32}$$

At this stage, multiplying the first equation of (1) by u_x^* and u_y^*, respectively, and considering their complex conjugates we have

$$\begin{aligned}\mathrm{i}(u_tu_x^* - u_t^*u_x) &= -(|u_x|^2)_x - (u_x^*u_{yy} + u_xu_{yy}^*) + \chi|u|^2(|u|^2)_x + b(\varphi_{1,x} + \varphi_{2,y})(|u|^2)_x,\\ \mathrm{i}(u_tu_y^* - u_t^*u_y) &= -(|u_y|^2)_y - (u_y^*u_{xx} + u_yu_{xx}^*) + \chi|u|^2(|u|^2)_y + b(\varphi_{1,x} + \varphi_{2,y})(|u|^2)_y.\end{aligned} \tag{33}$$

Using equations (33) in equation (32), after performing several integration by parts we obtain

$$\frac{\mathrm{d}^2I}{\mathrm{d}t^2} = 8\int_{\mathbb{R}^2} \left\{|u_x|^2 + |u_y|^2 + \frac{\chi}{2}|u|^4\right\}\mathrm{d}x\,\mathrm{d}y + 4b(J_1 + J_2), \tag{34}$$

where J_1 and J_2 are given by

$$\begin{aligned}J_1 &= \int_{\mathbb{R}^2} \varphi_{1,x}[2|u|^2 + x(|u|^2)_x + y(|u|^2)_y]\,\mathrm{d}x\,\mathrm{d}y,\\ J_2 &= \int_{\mathbb{R}^2} \varphi_{2,y}[2|u|^2 + x(|u|^2)_x + y(|u|^2)_y]\,\mathrm{d}x\,\mathrm{d}y.\end{aligned} \tag{35}$$

Using the second equation of the GDS system (1) in equation $(35)_1$ and integrating by parts, after lengthy but straightforward calculations, we obtain J_1 as

$$J_1 = \int_{\mathbb{R}^2} [(\varphi_{1,x})^2 + m_2(\varphi_{1,y})^2 + \frac{n}{2}(\varphi_{1,y}\varphi_{2,x} + \varphi_{1,x}\varphi_{2,y})]\,\mathrm{d}x\,\mathrm{d}y + n\int_{\mathbb{R}^2}(x\varphi_{1,x}\varphi_{2,xy} + y\varphi_{1,y}\varphi_{2,xy})\,\mathrm{d}x\,\mathrm{d}y.$$

Similarly, using the third equation of (1) in equation $(35)_2$ and integrating by parts, we obtain J_2 as

$$J_2 = \int_{\mathbb{R}^2} [\lambda(\varphi_{2,x})^2 + m_1(\varphi_{2,y})^2 + \frac{n}{2}(\varphi_{1,y}\varphi_{2,x} + \varphi_{1,x}\varphi_{2,y})]\,\mathrm{d}x\,\mathrm{d}y - n\int_{\mathbb{R}^2}(x\varphi_{1,x}\varphi_{2,xy} + y\varphi_{1,y}\varphi_{2,xy})\,\mathrm{d}x\,\mathrm{d}y.$$

Finally, substituting J_1 and J_2 in equation (34) results in

$$\begin{aligned}\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} &= 8\int_{\mathbb{R}^2}\Big\{|u_x|^2 + |u_y|^2 + \frac{\chi}{2}|u|^4 + \frac{b}{2}[(\varphi_{1,x})^2 + m_2(\varphi_{1,y})^2 + \lambda(\varphi_{2,x})^2 + m_1(\varphi_{2,y})^2 \\ &\quad + n(\varphi_{1,y}\varphi_{2,x} + \varphi_{1,x}\varphi_{2,y})]\Big\}\,\mathrm{d}x\,\mathrm{d}y, \\ &= 8H. \end{aligned} \tag{36}$$

Thus, we have proved the following lemma.

Lemma 6.1. *Suppose that* $(u, \varphi_1, \varphi_2)$ *be a solution of the GDS system (1) satisfying that* $u \in H^1(\mathbb{R}^2)$, $\nabla\varphi_1, \nabla\varphi_2 \in \mathbb{L}^2(\mathbb{R}^2)$*, and let*

$$I(t) = \int_{\mathbb{R}^2}(x^2 + y^2)|u|^2\,\mathrm{d}x\,\mathrm{d}y,$$

then

$$\frac{\mathrm{d}I}{\mathrm{d}t} = 4\,\mathrm{Im}\int_{\mathbb{R}^2}(xu^*u_x + yu^*u_y)\,\mathrm{d}x\,\mathrm{d}y,$$

and

$$\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = 8H.$$

Lemma 6.2. *Suppose that* $m_1 > 1$ *and* $\chi < \min\left(-\frac{b}{m_1}, 0\right)$*, then there exists* $u_0 \in H^1(\mathbb{R}^2)$ *such that* $H(u_0) < 0$.

Proof. Let us consider the function $v_\mu(x, y) = \mu\exp\left(-\frac{x^2}{\beta^2} - \frac{y^2}{\gamma^2}\right)$ in $H^1(\mathbb{R}^2)$, where μ, β and γ are positive. Using the Hamiltonian obtained in (26), the energy of v_μ is written as

$$H(v_\mu) = \int\left(\xi_1^2 + \xi_2^2\right)|\hat{v}_\mu|^2\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2 + \frac{1}{2}\int(\chi + \alpha b)|\hat{f}_\mu|^2\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2 \tag{37}$$

where $\hat{v}_\mu$ is the Fourier transform of v_μ and $\hat{f}_\mu$ is the Fourier transform of $|v_\mu|^2$, they are given by

$$\begin{aligned}\hat{v}_\mu &= \frac{\mu\beta\gamma}{2}\exp\left(-\frac{\beta^2\xi_1^2}{4} - \frac{\gamma^2\xi_2^2}{4}\right), \\ \hat{f}_\mu &= \frac{\mu^2\beta\gamma}{4}\exp\left(-\frac{\beta^2\xi_1^2}{8} - \frac{\gamma^2\xi_2^2}{8}\right).\end{aligned} \tag{38}$$

Substituting results (38) into the energy (37) gives

$$H(v_\mu) = K_1\mu^2 + K_2\mu^4,$$

where

$$K_1 = \frac{\beta^2\gamma^2}{4}\int \left(\xi_1^2+\xi_2^2\right)\exp\left[\frac{1}{2}\left(-\beta^2\xi_1^2-\gamma^2\xi_2^2\right)\right]\mathrm{d}\xi_1\,\mathrm{d}\xi_2,$$

$$K_2 = \frac{\beta^2\gamma^2}{32}\int (\chi+\alpha b)\exp\left[\frac{1}{4}\left(-\beta^2\xi_1^2-\gamma^2\xi_2^2\right)\right]\mathrm{d}\xi_1\,\mathrm{d}\xi_2,$$

where α is given in (27). By using the following transformations:

$$\xi_1 = \gamma r\cos\theta, \qquad \xi_2 = \beta r\sin\theta, \qquad \mathrm{d}\xi_1\,\mathrm{d}\xi_2 = \gamma\beta r\,\mathrm{d}r\,\mathrm{d}\theta,$$

the integrals K_1 and K_2 are obtained as

$$K_1 = \frac{\beta^3\gamma^3}{4}\int_0^\infty \exp\left(\frac{-\beta^2\gamma^2r^2}{2}\right)r^3\,\mathrm{d}r\int_0^{2\pi}(\gamma^2\cos^2\theta+\beta^2\sin^2\theta)\,\mathrm{d}\theta,$$

$$K_2 = \frac{\beta^3\gamma^3}{32}\int_0^\infty \exp\left(\frac{-\beta^2\gamma^2r^2}{4}\right)r\,\mathrm{d}r \times\int_0^{2\pi}\left(\chi+b\frac{\lambda\gamma^4\cos^4+(1+m_1-2n)\gamma^2\beta^2\sin^2\theta\cos^2\theta+m_2\beta^4\sin^4\theta}{(\lambda\gamma^2\cos^2\theta+m_2\beta^2\sin^2\theta)(\gamma^2\cos^2\theta+m_1\beta^2\sin^2\theta)}\right)\mathrm{d}\theta.$$

After some long but straightforward calculations, we get

$$H(v_\mu) = \frac{\pi(\beta^2+\gamma^2)}{2\beta\gamma}\mu^2 + \frac{\pi\beta\gamma}{8}(\chi+Jb)\mu^4, \tag{39}$$

where

$$J = \frac{(\lambda m_1+\sqrt{\lambda m_1m_2})\gamma^2+[\sqrt{m}_1(1-2n+m_1)+\sqrt{\lambda m_2}(1+m_1)]\beta\gamma+(m_2+\sqrt{\lambda m_1m_2})\beta^2}{(m_1\sqrt{\lambda}+\sqrt{m_1m_2})[\sqrt{\lambda}\gamma^2+(\sqrt{m}_2+\sqrt{\lambda m_1})\beta\gamma+\sqrt{m_1m_2}\beta^2]}. \tag{40}$$

If $m_1 > 1$ then $0 < \frac{1}{m_1} < J < 1$ (see appendix B). Let $\chi < \min\left(-\frac{b}{m_1},0\right)$. If $b < 0$ then $\chi < 0$. This leads to $\chi + Jb < 0$. On the other hand, if $b \geqslant 0$ then $\chi + \frac{b}{m_1} < 0$. Then, it is possible to choose J arbitrarily close to $\frac{1}{m_1}$ by selecting the parameters β and γ appropriately. Therefore, if $\chi < \min\left(-\frac{b}{m_1},0\right)$ then by suitably choosing the arbitrary positive μ, we conclude that there exist solutions $v_\mu(x,y)$ that make the energy (39) negative. □

Theorem 6.1. *Let u be the solution of the Cauchy problem for the GDS system (1) with the initial value* $u_0 \in H^1(\mathbb{R}^2)$ *and* $\nabla\varphi_1, \nabla\varphi_2 \in \mathbb{L}^2(\mathbb{R}^2)$ *and* $u \in H^1(\mathbb{R}^2)$. *If one of the following conditions holds:*

(i) $H(u_0) < 0$,
(ii) $H(u_0) = 0$ *and* $\operatorname{Im}\int_{\mathbb{R}^2}(xu_0^*u_{0,x}+yu_0^*u_{0,y})\,\mathrm{d}x\,\mathrm{d}y < 0$,
(iii) $H(u_0) > 0$ *and* $-\operatorname{Im}\int_{\mathbb{R}^2}(xu_0^*u_{0,x}+yu_0^*u_{0,y})\,\mathrm{d}x\,\mathrm{d}y \geqslant \sqrt{H(u_0)I(u_0)}$,

then

$$\lim_{t\to T^-}(\|u_x\|_{L^2}+\|u_y\|_{L^2}) = \infty,$$

that is, if the solution exists for all finite time then it will blow up in finite time.

Proof. Throughout, we will assume that solutions exist as long as they remain bounded in H^1-norm. According to (36), we have

$$I(t) = \int_{\mathbb{R}^2}(x^2+y^2)|u|^2\,\mathrm{d}x\,\mathrm{d}y = 4H(u_0)t^2 + I'(u_0)t + \int_{\mathbb{R}^2}(x^2+y^2)|u_0|^2\,\mathrm{d}x\,\mathrm{d}y. \tag{41}$$

From the conservation of mass (17), we write

$$\begin{aligned}\|u_0\|_{L^2}^2 &= \|u\|_{L^2}^2\\ &= -\frac{1}{2}\int_{\mathbb{R}^2} x(uu^*)_x\,\mathrm{d}x\,\mathrm{d}y - \frac{1}{2}\int_{\mathbb{R}^2} y(uu^*)_y\,\mathrm{d}x\,\mathrm{d}y,\\ &= -\mathrm{Re}\int_{\mathbb{R}^2}(xu_xu^* + yu_yu^*)\,\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{42}$$

By the Cauchy–Schwartz inequality from (42) we have

$$\|u_0\|_{L^2}^2 \leqslant \|xu\|_{L^2}\|u_x\|_{L^2} + \|yu\|_{L^2}\|u_y\|_{L^2}. \tag{43}$$

Since

$$\|xu\|_{L^2}^2 \leqslant I(t), \qquad \|yu\|_{L^2}^2 \leqslant I(t),$$

it follows from (43) that

$$\|u_0\|_{L^2}^2 \leqslant \sqrt{I(t)}(\|u_x\|_{L^2} + \|u_y\|_{L^2}).$$

Therefore, in the case of $I(T) = 0$, H^1-norm of u becomes unbounded, i.e. the solution of the GDS system will blow up in finite time.

(i) By lemma 6.2 there exist some solutions u_0 such that $H(u_0) < 0$. Thus, when $H(u_0) < 0$ for a sufficiently large time $I(T) = 0$, i.e. the solution will blow up in the finite time T.

(ii) If $H(u_0) = 0$ and

$$I'(u_0) = 4\,\mathrm{Im}\int_{\mathbb{R}^2}(xu_0^*u_{0,x} + yu_0^*u_{0,y})\,\mathrm{d}x\,\mathrm{d}y < 0,$$

then in the finite time, $I(T) = 0$.

(iii) In the case of $H(u_0) > 0$, it is possible to show that there exists T such that $I(T) = 0$ if

$$\begin{aligned}&-I'(u_0) \geqslant 4\sqrt{H(u_0)I(u_0)},\\ &-\mathrm{Im}\int_{\mathbb{R}^2}(xu_0^*u_{0,x} + yu_0^*u_{0,y})\,\mathrm{d}x\,\mathrm{d}y \geqslant \sqrt{H(u_0)I(u_0)}.\end{aligned}$$

□

Remark. Similar global nonexistence results for the NLS equation and a degenerate DS system have been given by Weinstein [17] and Li, Guo and Jiang [18], respectively (see also [19]).

In order to justify rigorously the blow-up of solutions as stated in theorem 6.1, one would need to establish the local in time existence of solutions in the weighted $L_w^2(\mathbb{R}^2)$ spaces with $w(x, y) = x^2 + y^2$ as well as a continuation principle for H^1-bounded solutions (see section 7).

For GDS, as it is the case for NLS and DS, the global nonexistence can be interpreted both as a singularity at $(x, y) = (0, 0)$ for the solution u and as the unboundedness of the 'energy', i.e H^1-norm, of u. The latter sense is the content of theorem 6.1. Here, let us briefly remark on the singularity formation. Since, $\lim_{t\to T^-} I(t) = 0$, it follows that for all $\epsilon > 0$, $\int_{|x|>\epsilon}|u|^2\,\mathrm{d}x\,\mathrm{d}y = 0$ hence $M(u_0) = M(u) = \int_{|x|<\epsilon}|u|^2\,\mathrm{d}x\,\mathrm{d}y$ at $t = T$. By the mean value theorem for integrals it follows that there exists $(x_\epsilon, y_\epsilon) \to (0, 0)$ as $\epsilon \to 0$ such that $\lim_{\epsilon\to 0}|u(x_\epsilon, y_\epsilon, T)| = +\infty$ (see also [20]).

Similar to the case of NLS equation [17], an immediate result of theorem 6.1 is the following instability of nonzero ground states in the elliptic–elliptic–elliptic case.

Lemma 6.3. *The nontrivial positive H^1 ground state solutions of the GDS system (1) are unstable.*

Proof. We will assume that there exists a nonzero ground state $R : \mathbb{R}^2 \to \mathbb{R}^+$ such that

$$\begin{aligned} \Delta R - R &= \chi R^3 + b(\varphi_{1,x} + \varphi_{2,y})R, \\ \varphi_{1,xx} + m_2\varphi_{1,yy} + n\varphi_{2,xy} &= (R^2)_x, \\ \lambda\varphi_{2,xx} + m_1\varphi_{2,yy} + n\varphi_{1,xy} &= (R^2)_y, \end{aligned} \tag{44}$$

i.e. $R = R(x, y)$, and $\varphi_1 = \varphi_1(x, y)$ and $\varphi_2 = \varphi_2(x, y)$ so that $u = \mathrm{e}^{\mathrm{i}t} R$ and φ_1, φ_2 solve (1). Then $I(t) = \int_{\mathbb{R}^2}(x^2 + y^2)R^2(x, y)\,\mathrm{d}x\,\mathrm{d}y$ is time independent and $8H(R) = \frac{\mathrm{d}^2}{\mathrm{d}t^2} I(t) = 0$. In the elliptic case,

$$0 = H(R) = \int \left(\xi_1^2 + \xi_2^2\right)|\hat{R}(\xi)|^2\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2 + \frac{1}{2}\int (\chi + \alpha b)|\hat{F}(\xi)|^2\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2,$$

where $\hat{F}$ is the Fourier transform of R^2, since the first term is positive, it follows that

$$-\int \left(\xi_1^2 + \xi_2^2\right)|\hat{R}|^2\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2 = \frac{1}{2}\int (\chi + \alpha b)|\hat{F}|^2\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2 < 0. \tag{45}$$

Note that here α depends homogeneously on (ξ_1, ξ_2). Then

$$H[(1+\epsilon)R] = (1+\epsilon)^2 \int \left(\xi_1^2 + \xi_2^2\right)|\hat{R}|^2\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2 + \frac{1}{2}(1+\epsilon)^4 \int (\chi + \alpha b)|\hat{F}|^2\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2,$$

since $\hat{F} = (1+\epsilon)^2\hat{R}^2$. Using (45)

$$H[(1+\epsilon)R] = [(1+\epsilon)^2 - (1+\epsilon)^4] \int \left(\xi_1^2 + \xi_2^2\right)|\hat{R}|^2\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2.$$

By theorem 6.1 when $H(u_0) < 0$ then the solution blows up, stating that the ground state of the GDS system is unstable. Note that we have made no assumptions on m_1, χ and b. □

7. Concluding remarks

As mentioned before the arguments furnished in this work hinge upon various existence and uniqueness theorems for the solutions. A complete and detailed discussion of an existence and uniqueness result, which is suitable for our purposes, and a result on the continuation of solutions is beyond the scope of this work. However, an argument parallel to the one given in [4], namely the proof of theorem 2.2 (pp 483–8), applies with minor modifications. This observation is based on the fact that both DS and GDS equations can be posed as an NLS equation with an additional non-local term. In the case of DS equations, which is given as

$$\mathrm{i}u_t + \delta u_{xx} + u_{yy} = \chi|u|^2 u + buE(|u|^2)$$

in [4] (equation (2.5)), the non-local term is expressed in terms of transformed variables as

$$\widehat{E(f)}(\xi_1, \xi_2) = \frac{\xi_1^2}{m\xi_1^2 + \xi_2^2}\hat{f}(\xi_1, \xi_2) = p(\xi_1, \xi_2)\hat{f}(\xi_1, \xi_2). \tag{46}$$

Whereas, in our case, equations (1) can be rewritten as

$$\mathrm{i}u_t + u_{xx} + u_{yy} = \chi|u|^2 u + bK(|u|^2)u, \tag{47}$$

where the new non-local term K is given in terms of transformed variables as

$$\widehat{K(f)}(\xi_1, \xi_2) = \alpha(\xi_1, \xi_2)\hat{f}(\xi_1, \xi_2) \tag{48}$$

with α as given in (27).

Our first observation is that like E, K is a bounded linear operator from $L^p(\mathbb{R}^2)$ into itself for $1 < p < \infty$. In fact, similar to the symbol $p(\xi)$ given in (46), $\alpha(\xi)$ is also a homogeneous symbol of order 0; moreover, by results in appendix A, for $m_1 > 0$,

$0 < \alpha(\xi_1, \xi_2) \leqslant \max\left(1, \frac{1}{m_1}\right) \equiv \alpha_M$. It follows from Calderon–Zygmund theorem ([21], theorem 5.16) that $K : L^p \to L^p$ is a bounded operator, and $\|K(f)\|_{L^2} \leqslant \alpha_M \|f\|_{L^2}$ for $p = 2$.

Our second observation is along the lines of the argument given in [4], for Theorem 2.2 there. For $u_0 \in H^1(\mathbb{R}^2)$, the existence of a solution to the Cauchy problem for (1) is equivalent to the existence of the fixed point for the map

$$(\mathcal{T}v)(t) = S(t)u_0 - \mathrm{i}\int_0^t S(t-s)F(v(s))\,\mathrm{d}s \tag{49}$$

where the nonlinear term F is given by

$$F(v) = \chi|v|^2 v + bvK(|v|^2) \tag{50}$$

and where $S(t)$ represents the solution semi-group for the linear Schrödinger equation. In [4], following the argument given in [22] for local existence and uniqueness of solutions of NLS, i.e. proof of theorem 1, it is shown that a map, similar to $\mathcal{T}$ in (49), is a contraction from $B_R(u_0, Y)$ into itself with respect to X-norm when T is chosen small enough. The spaces X and Y are given in [22], as well as in [4], as

$$\begin{aligned} X &= L^\infty([0,T]; L^2(\mathbb{R}^2)) \cap L^4([0,T]; L^4(\mathbb{R}^2)) \\ Y &= \{v \in X : \nabla v \in X\} \end{aligned} \tag{51}$$

with the corresponding norms

$$\|v\|_X = \max\{\|v\|_{\infty,2}, \|v\|_{4,4}\}, \qquad \|v\|_Y = \max\{\|v\|_X, \|\nabla v\|_X\}, \tag{52}$$

where we have used the notation $\|v\|_{p,q} \equiv$ the $L^p([0,T]; L^q(\mathbb{R}^2))$ norm of v.

The set $B \equiv B_R(u_0, Y)$ is complete in X with respect to the X-norm given in (51) and in order to apply the contraction mapping principle to $\mathcal{T}$ acting on B, we must show that $\mathcal{T}\mathcal{B} \subseteq \mathcal{B}$ and $\mathcal{T}$ is a strict contraction for a suitably chosen T. To adopt the argument in [22] to the present setting we proceed as in [4] to estimate the contribution of the non-local term $buK(|u|^2)$ to the nonlinearity $F(u)$ in equation (50). Letting $\vec{\varphi} = (\varphi_1, \varphi_2)$, we see that

$$K(|u|^2) = \operatorname{div}\vec{\varphi}, \tag{53}$$

and one can replace in the argument starting from p 485 in [4] φ_x by $\operatorname{div}\vec{\varphi}$ and proceed in an identical manner. For example,

$$\|\operatorname{div}\vec{\varphi}\|_{\infty,2} = \|K(|u|^2)\|_{\infty,2} \leqslant \alpha_M \|u\|^2_{\infty,4} \tag{54}$$

by the (L^2, L^2)-bound on the operator K. Similarly, since K is linear,

$$\|\operatorname{div}(\vec{\varphi} - \vec{\psi})\|_{2,2} = \|K(|u|^2 - |v|^2)\|_{2,2} \leqslant \alpha_M \||u|^2 - |v|^2\|_{2,2}. \tag{55}$$

Consequently, one can show as in (2.18) in [4] that

$$\mathrm{Lip}_X(\mathcal{T}) \leqslant CT^{\frac{1}{2}}(R + \|u_0\|_{H^1(\mathbb{R}^2)})^2 \tag{56}$$

where C is independent of R, T and $\|u_0\|_{H^1(\mathbb{R}^2)}$. By choosing T small enough one can guarantee that $\mathcal{T}B \subseteq B$ and it is a strict contraction. This choice depends only on $\|u_0\|_{H^1(\mathbb{R}^2)}$, which leads us to our final observation on the continuation of solutions; as long as $\|u(t)\|_{H^1(\mathbb{R}^2)}$ remains bounded. Much in the spirit of theorem 3.4.1 in [19] (pp 74–5) a new solution starting from $u(T)$ can be added to a solution on $[0, T)$ to give a new solution on $[0, 2T)$. Since $\|u(T)\|^2_{H^1(\mathbb{R}^2)} \leqslant H(u_0) + \|u_0\|^2_{L^2(\mathbb{R}^2)}$ as in theorem 5.1, choosing T so as to make $CT^{\frac{1}{2}}\left\{R^2 + \|u_0\|^2_{L^2(\mathbb{R}^2)} + H(u_0)\right\} < 1$, the maximal interval of the existence of solutions can be extended (condition on T to assure $\mathcal{T}B \subseteq B$ is treated similarly).

The argument outlined above gives a unique maximal solution for the Cauchy problem for the GDS system that is in $C([0, T); H^1(\mathbb{R}^2)) \cap C^1((0, T); H^{-1}(\mathbb{R}^2))$, which is similar to theorem I in [22] for NLS and theorem 2.1 in [23] for DS system.

The line of reasoning given above when furnished with necessary details has two-fold impact on the results given in this paper. Firstly, it justifies the conclusion of theorem 5.1 for the global existence of solutions, since whenever $\chi \geqslant \max\left\{-b \max\left(1, \frac{1}{m_1}\right), 0\right\}$ H^1-norm of $u(t)$ is controlled by $H(u_0)$ and L^2-norm of u_0 for every t; therefore, the maximal interval of existence is $[0, \infty)$. Secondly, it suggests that our global nonexistence result can be a bona fide blow-up result since as long as H^1-norm is controlled the solution can be extended. Namely, let $t = T^*$ be the end of the maximal interval of the existence and $\lim_{t\to T^{*-}} \|u(t)\|_{H^1(\mathbb{R}^2)} < \infty$. For $R = \sup_{0\leqslant t<T^*} \|u(t)\|^2_{H^1(\mathbb{R}^2)}$, any solution can be extended by T_1 where $4CT_1^{\frac{1}{2}}R^2 < 1$. Hence T_1 is independent of t no matter how close t is to T^* (here again the additional condition on T_1 so that $\mathcal{T}B \subseteq B$ is handled similarly).

In the present study, we have shown that the solutions of the GDS system in the elliptic–elliptic–elliptic case will exist globally if they exist local in time provided that the coefficients of the nonlinear terms satisfy certain conditions, i.e. $\chi \geqslant \max\left\{-b \max\left(1, \frac{1}{m_1}\right), 0\right\}$ if $m_1 > 0$. As a consequence of the global existence, stability of the zero solution of the GDS system has been established under the same condition. We have also shown that the solutions of the GDS system cannot exist globally in the elliptic–elliptic–elliptic case using the virial theorem, if $\chi < \min\left\{-\frac{b}{m_1}, 0\right\}$. As a consequence of this result we show that the nontrivial ground state solutions of the GDS system are unstable.

As a final note, contrary to the DS system our global existence and nonexistence results, theorem 5.1 and lemma 6.2 in conjunction with theorem 6.1 respectively, do not cover the whole parameter range for χ and b. Assuming that $m_1 > 1$, for $\chi \geqslant \min\left\{-\frac{b}{m_1}, 0\right\}$ and $\chi < \max\{-b, 0\}$ both global existence and/or nonexistence of solutions remain an open problem.

Acknowledgments

We thank the referees for helping us to clarify some of the issues in this paper, especially pointing out the necessity of discussion on the continuation of H^1 solutions. The second author would also like to thank V Kalantarov for some illuminating discussions.

Appendix A

(a) If $m_1 \geqslant 1$ then $0 < \alpha \leqslant 1$. Since $n^2 = (\lambda - 1)(m_2 - m_1)$, by simple algebra we see that $2\sqrt{\lambda m_2} + 1 + m_1 - 2n$ is positive. Rewriting the numerator of (27) as

$$\lambda\xi_1^4 + (1 + m_1 - 2n)\xi_1^2\xi_2^2 + m_2\xi_2^4 = \left(\sqrt{\lambda}\xi_1^2 - \sqrt{m_2}\xi_2^2\right)^2 + (2\sqrt{\lambda m_2} + 1 + m_1 - 2n)\xi_1^2\xi_2^2,$$

the right-hand side of the above equality becomes positive, so $\alpha > 0$. When $m_1 \geqslant 1$,

$$m_1 + 1 - 2n < 1 + m_1 < m_2 + \lambda m_1.$$

But, whenever d, e and $f > 0$, we have

$$\min\left\{\frac{a}{d}, \frac{b}{e}, \frac{c}{f}\right\} \leqslant \frac{a+b+c}{d+e+f} \leqslant \max\left\{\frac{a}{d}, \frac{b}{e}, \frac{c}{f}\right\}.$$

So that

$$\alpha(\xi_1, \xi_2) \leqslant \max\left\{1, \frac{m_1 + 1 - 2n}{\lambda m_1 + m_2}, \frac{1}{m_1}\right\} \leqslant 1.$$

(b) If $0 < m_1 < 1$ then $1 \leqslant \alpha \leqslant \frac{1}{m_1}$. α can take any value between 1 and $\frac{1}{m_1}$ as a result of the intermediate value theorem.

Appendix B

It is straightforward to check that

$$\sqrt{m}_1(1 - 2n + m_1) + \sqrt{\lambda m_2}(1 + m_1) \geqslant 0$$

which implies that J as given in (40) is positive. On the other hand, for $m_1 > 1$, one can choose β large enough so that

$$[\sqrt{m}_1(1 - 2n + m_1) + \sqrt{\lambda m_2}(1 + m_1) - \sqrt{m}_1(\sqrt{\lambda m_1} + \sqrt{m}_2)^2]\beta\gamma + (1 - m_1)(m_2 + \sqrt{\lambda m_1 m_2})\beta^2$$

becomes negative. This in turn implies that $J < 1$. However, J cannot be made arbitrarily close to zero for any choice of β and γ. In fact, $0 < \frac{1}{m_1} < J < 1$ as a result of the intermediate value theorem.

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